Differentiation from First Principles 1

1.

Prove, from first principles, that the derivative of $6x$ is 6. **(3 marks)**

2.

Prove, from first principles, that the derivative of $4x^2$ is $8x$. **(4 marks)**

3.

$f(z) = az^2$, where a is a constant. Prove, from first principles, that $f'(z) = 2az$. **(4 marks)**

4.

$y = 3x^2$

a Work out $\frac{dy}{dx}$ from first principles.

b Calculate the gradient of the tangent where $x = 5$

5.

$y = x^3 - 2x$

a Work out $\frac{dy}{dx}$ from first principles.

b Calculate the gradient of the tangent where $x = 2$

6.

Differentiate from first principles

$$y = 4x^2 + x$$

[4 marks]

Challenge

$f(x) = \frac{1}{x}$

a Given that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, show that $f'(x) = \lim_{h \to 0} \frac{-1}{x^2 + xh}$

b Deduce that $f'(x) = -\frac{1}{x^2}$

Solutions

1.

$$\mathrm{f}(x) = 6x$$

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h} \\
&= \lim_{h\to 0}\frac{6(x+h)-6x}{h} \\
&= \lim_{h\to 0}\frac{6x+6h-6x}{h} \\
&= \lim_{h\to 0}\frac{6h}{h} \\
&= \lim_{h\to 0} 6
\end{aligned}$$

So $\mathrm{f}'(x) = 6$

2.

$$\mathrm{f}(x) = 4x^2$$

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h} \\
&= \lim_{h\to 0}\frac{4(x+h)^2-4x^2}{h} \\
&= \lim_{h\to 0}\frac{4x^2+8xh+4h^2-4x^2}{h} \\
&= \lim_{h\to 0}\frac{8xh+4h^2}{h} \\
&= \lim_{h\to 0}\frac{h(8x+4h)}{h} \\
&= \lim_{h\to 0}(8x+4h)
\end{aligned}$$

As $h \to 0$, $8x + 4h \to 8x$.

So $\mathrm{f}'(x) = 8x$

3.

$$f(z) = az^2$$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{a(z+h)^2 - az^2}{h}$$

$$= \lim_{h \to 0} \frac{az^2 + 2azh + ah^2 - az^2}{h}$$

$$= \lim_{h \to 0} \frac{2azh + ah^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2az + ah)}{h}$$

$$= \lim_{h \to 0} (2az + ah)$$

As $h \to 0$, $2az + ah \to 2az$.

So $f'(z) = 2az$

4.

a	$\frac{dy}{dx} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{(x+h) - x}$	M1	Using $x + h$ or $x + \delta x$
	$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$	M1	$\frac{\text{change in } y}{\text{change in } x}$
	$= \lim_{h \to 0} (6x + 3h)$	A1	$6x + 3h$
	$= 6x$	M1	Finding limit as $h \to 0$
		A1	
b	$x = 5 \Rightarrow$ gradient $= 6 \times 5$	M1	Substituting into $\frac{dy}{dx}$
	$= 30$	A1	

5.

a	$\frac{dy}{dx} = \lim_{h \to 0} \frac{[(x+h)^3 - 2(x+h)] - [x^3 - 2x]}{(x+h) - x}$	M1	Using $x + h$ or $x + \delta x$
		M1	$\frac{\text{change in } y}{\text{change in } x}$
	$= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h)] - [x^3 - 2x]}{h}$	A1	$3x^2 + 3xh + h^2 - 2$
	$= \lim_{h \to 0}([3x^2 + 3xh + h^2 - 2)])$	M1	Finding limit as $h \to 0$
	$= 3x^2 - 2$	A1	
b	$x = 2 \Rightarrow$ gradient $= 3(2)^2 - 2$	M1	Substituting into $\frac{dy}{dx}$
	$= 10$	A1	

6.

Q	Marking Instructions	AO	Marks	Typical Solution
5	Uses correct formula and notation for this function; must have substituted $(x+h)$ correctly	1.1a	M1	$\lim_{h \to 0} \frac{4(x+h)^2 + (x+h) - (4x^2 + x)}{h}$
	Multiplies out $4(x+h)^2$ correctly	1.1b	B1	$\lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 + x + h - 4x^2 - x}{h}$
	Obtains numerator with no x^2 or x terms PI	1.1b	A1	$\lim_{h \to 0} \frac{8xh + 4h^2 + h}{h}$
	Completes rigorous argument, including dividing by h and correctly using limit	2.1	R1	$\lim_{h \to 0} (8x + 4h + 1)$ $= 8x + 1$
	Total		**4**	

Challenge

a $\mathrm{f}(x) = \frac{1}{x}$

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x^2 + hx}$$

b As $h \to 0$, $\frac{-1}{x^2 + hx} \to -\frac{1}{x^2}$.

So $\mathrm{f}'(x) = -\frac{1}{x^2}$